Option Pricing Model Based on a Markov-modulated Diffusion with Jumps

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Abstract

The paper proposes a class of financial market models which are based on inhomogeneous telegraph processes and jump diffusions with alternating volatilities. It is assumed that the jumps occur when the tendencies and volatilities are switching. We argue that such a model captures well the stock price dynamics under periodic financial cycles. The distribution of this process is described in detail. For this model we obtain the structure of the set of martingale measures. This incomplete model can be completed by adding another asset based on the same sources of randomness. Explicit closed-form formulae for prices of the standard European options are obtained for the completed market model.

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1 Introduction

Beginning with the works of Mandelbrot (1963), Mandelbrot and Taylor (1967), and Clark (1973), it is commonly accepted that the dynamics of asset returns cannot be described by geometric Brownian motion with constant parameters of drift and volatility. A lot of sophisticated constructions have been exploited to capture the features that help to express the reality better than Black-Scholes-Merton model. Merton (1976) which have incorporated jump diffusion model for the asset price was the first. Later on the constructions with random drift and random volatility parameters appeared. A popular approach is to use Lévy processes with stationary independent increments. However, this theoretical behavior does not match empirical observations.

Another approach utilizes markovian dependence on the past and the technique of Markov random processes (see Elliott and van der Hoek (1997)). We deal mainly with this direction. More precisely, the model is based on a standard Brownian motion $w = w(t), t \geq 0$ and on a Markov process $\varepsilon(t), t \geq 0$ with two states $0, 1$ and transition probability intensities λ_0 and λ_1.

Let us define processes $c_{\varepsilon(t)}$, $\sigma_{\varepsilon(t)}$ and $r_{\varepsilon(t)}, t \geq 0$, where $c_0 \geq c_1, r_0, r_1 > 0$. Then, we introduce $\mathcal{T}(t) = \int_0^t c_{\varepsilon(\tau)}\mathrm{d}\tau$, $\mathcal{D}(t) = \int_0^t \sigma_{\varepsilon(\tau)}\mathrm{d}w(\tau)$ and a pure jump process $\mathcal{J} = \mathcal{J}(t)$ with alternating jumps of sizes h_0 and h_1, $h_0, h_1 > -1$.

The continuous time random motion $\mathcal{T}(t) = \int_0^t c_{\varepsilon(\tau)}\mathrm{d}\tau, t \geq 0$ with alternating velocities is known as telegraph process. This type of processes have been used before in various

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probabilistic aspects (see, for instance, Goldstein (1951), Kac (1974) and Zacks (2004)). These processes have been exploited for stochastic volatility modeling (Di Masi et al (1994)), as well as for obtaining a "telegraph analog" of the Black-Scholes model (Di Crescenzo and Pellerey (2002)). The option pricing models based on continuous-time random walks are widely presented in the physics literature (see Masoliver et al (2006) or Montero (2008)). Recently the telegraph processes was applied to actuarial problems, Mazza and Rullière (2004). Markov-modulated diffusion process $\mathcal{D}(t) = \int_0^t \sigma_{\varepsilon(\tau)} \mathrm{d}w(\tau)$ was exploited for financial market modeling (see Guo (2001), Jobert and Rogers (2006)), as well as in insurance (see Bäuerle and Kötter (2007)) or in theory of queueing networks (see Ren and Kobayashi (1998)).

This paper deals with the market model which presumes the evolution of risky asset $S(t)$ is given by the stochastic exponential of the sum $\mathcal{X} = \mathcal{T}(t) + \mathcal{D}(t) + \mathcal{J}(t)$. The bond price is the usual exponential of the process $\mathcal{Y} = \mathcal{Y}(t) = \int_0^t r_{\varepsilon(\tau)} \mathrm{d}\tau, t \geq 0$ with alternating interest rates r_0 and r_1.

This model generalizes classic Black-Scholes-Merton model based on geometric Brownian motion ($c_0 = c_1,\ r_0 = r_1,\ \sigma_0 = \sigma_1 \neq 0,\ h_0 = h_1 = 0$), Black and Scholes (1973), Merton (1973). Other particular versions of this model was also discussed before:

1. $c_0 = c_1,\ \sigma_0 = \sigma_1 = 0,\ h_0 = h_1 \neq 0$: *Merton model*, Merton (1976), Cox and Ross (1976);

2. $c_0 \neq c_1,\ \sigma_0 = \sigma_1 = 0,\ h_0 \neq h_1$: *jump-telegraph model*, Ratanov (2007);

3. $c_0 \neq c_1,\ \sigma_0 \neq \sigma_1,\ h_0 = h_1 = 0$: *Markov-modulated dynamics*, Guo (2001), Jobert and Rogers (2006).

The jump-telegraph model, as well as Black-Scholes and Merton model, is free of arbitrage opportunities, and it is complete. Moreover it permits explicit standard option pricing formulae similar to the classic Black-Scholes formula. Under suitable rescaling this model converges to the Black-Scholes (see Ratanov (2007)). First calibration results of the parameters of the telegraph model have been presented in De Gregorio and Iacus (2007). These estimations have been based on the data of Dow-Jones industrial average (July 1971 - Aug 1974). However, a presence of jumps and/or diffusion components has not been estimated. Nevertheless, an implied volatility with respect to a moneyness variable in stochastic volatility models of the Ornstein-Uhlenbeck type (see Nicolato and Venardos (2003)) looks very similar to the volatility smile in jump telegraph model (see Ratanov (2007b)).

In this paper we extend the jump-telegraph market model, presented in Ratanov (2007), by adding the diffusion component with alternating volatility coefficient.

The jump-telegraph model equipped with the diffusion term becomes more realistic. Indeed, the alternating velocities of the telegraph process describe long-term financial trends, and the diffusion summand introduces an uncertainty of current prices. This uncertainty may has different volatilities in the bearish and in the bullish trends ($\sigma_0 \neq \sigma_1$).

The paper is organized as follows: in Section 2 we present the detailed definitions and the description of underlying processes and their distributions. The explicit construction of a measure change is given by the Girsanov theorem for jump telegraph-diffusion processes.

In Section 3 we describe the set of risk-neutral measures for the incomplete jump telegraph-diffusion model. Also we consider a completion of the model by adding another

asset driven by the same sources of randomness. For the completed market model we obtain explicit option pricing formulae of the standard call option. These formulae are based on a mix of Black-Scholes function and densities of spending times of the driving Markov flow.

2 Jump telegraph processes and jump diffusions with Markov switching

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. Denote $\varepsilon_i(t),\ t \geq 0,\ i = 0, 1$ a pair of Markov processes with two states $\{0, 1\}$ and with rates $\lambda_0, \lambda_1 > 0$:

$$\mathbb{P}\{\varepsilon_i(t + \Delta t) = j \mid \varepsilon_i(t) = j\} = 1 - \lambda_j \Delta t + o(\Delta t), \quad \Delta t \to 0,\ i, j = 0, 1.$$

Subscript i indicates the initial state: $\varepsilon_i(0) = i$.

Let $\tau_1, \tau_2, \ldots$ be switching times. The time intervals $\tau_j - \tau_{j-1},\ j = 1,\ 2,\ \ldots\ (\tau_0 = 0)$, separated by moments of value changes $\tau_j = \tau_j^i$ are independent and exponentially distributed. We denote by $\mathbb{P}_i$ the conditional probability with respect to the initial state $i = 0, 1$, and by $\mathbb{E}_i$ the expectation with respect to $\mathbb{P}_i$.

Denote by $N_i(t) = \max\{j : \tau_j \leq t\}, t \geq 0$ a number of switchings of ε_i till time $t,\ t \geq 0$. It is clear that $N_i,\ i = 0, 1$ are the counting Poisson processes with alternating intensities $\lambda_0, \lambda_1 > 0$. It is easy to see that the distributions $\pi_i(t; n) := \mathbb{P}_i\{N_i(t) = n\}, n = 0, 1, 2, \ldots,\ i = 0, 1,\ t \geq 0$ of the processes $N_i = N_i(t)$ satisfy the following system:

$$\begin{aligned} \frac{\mathrm{d}\pi_i(t; n)}{\mathrm{d}t} &= -\lambda_i \pi_i(t; n) + \lambda_i \pi_{1-i}(t; n-1), \quad i = 0, 1,\ n \geq 1, \\ \pi_i(t; 0) &= \mathrm{e}^{-\lambda_i t}. \end{aligned} \tag{2.1}$$

To prove it notice that conditioning on the Poisson event on the time interval $(0, \Delta t)$ one can obtain

$$\pi_i(t + \Delta t; n) = (1 - \lambda_i \Delta t)\pi_i(t; n) + \lambda_i \Delta t \pi_{1-i}(t; n-1) + o(\Delta t), \quad \Delta t \to 0,$$

which immediately leads to (2.1).

Let $c_0, c_1,\ c_0 > c_1;\ h_0, h_1; \sigma_0, \sigma_1$ be real numbers. Let $w = w(t),\ t \geq 0$ be a standard Brownian motion independent of ε_i. We consider

$$\mathcal{T}_i(t) = \mathcal{T}_i(t; c_0, c_1) = \int_0^t c_{\varepsilon_i(\tau)} \mathrm{d}\tau, \qquad \mathcal{J}_i(t) = \mathcal{J}_i(t; h_0, h_1) = \int_0^t h_{\varepsilon_i(\tau)} \mathrm{d}N_i(\tau) = \sum_{j=1}^{N_i(t)} h_{\varepsilon_i(\tau_j -)},$$

$$\mathcal{D}_i(t) = \mathcal{D}_i(t; \sigma_0, \sigma_1) = \int_0^t \sigma_{\varepsilon_i(\tau)} \mathrm{d}w(\tau). \tag{2.2}$$

Processes $\mathcal{T}_0, \mathcal{T}_1$ are telegraph processes with the states $< c_0, \lambda_0 >$ and $< c_1, \lambda_1 >$, $\mathcal{J}_0, \mathcal{J}_1$ have a sense of pure jump processes, and $\mathcal{D}_0, \mathcal{D}_1$ are Markov-modulated diffusions. Thus the sum $\mathcal{X}_i := \mathcal{T}_i(t) + \mathcal{J}_i(t) + \mathcal{D}_i(t),\ t \geq 0,\ i = 0, 1$ is naturally called jump telegraph-diffusion (JTD) process with two states, $< c_0, h_0, \sigma_0, \lambda_0 >$ and $< c_1, h_1, \sigma_1, \lambda_1 >$.

Further, we will assume all processes to be adapted to the natural filtration $\mathfrak{F}^i = (\mathfrak{F}^i_t)_{t\geq 0}$ ($\mathfrak{F}^i_0 = \{\emptyset,\ \Omega\}$), generated by $\varepsilon_i(t),\ t \geq 0$, and $w(t),\ t \geq 0$. We suppose that the filtration satisfies the "usual conditions" (see e. g. Karatzas and Schreve (1998)).

The distribution of $\mathcal{X}_i(t)$ can be found exactly. First, we denote by $p_i(x, t; n)$ (generalized) probability densities with respect to the measure $\mathbb{P}_i$ of the jump telegraph-diffusion variable $\mathcal{X}_i(t)$, which has n turns up to time t:

$$\mathbb{P}_i\{\mathcal{X}_i(t) \in \Delta,\ N_i(t) = n\} = \int\limits_{\Delta} p_i(x, t; n)\mathrm{d}x, \quad i = 0, 1,\ t \geq 0,\ n = 0, 1, 2, \dots \tag{2.3}$$

The PDEs which describe the densities $p_i(x, t; n)$ have the following form.

Theorem 2.1. *Densities $p_i, i = 0, 1$ satisfy the following PDE-system*

$$\frac{\partial p_i}{\partial t}(x, t; n) + c_i\frac{\partial p_i}{\partial x}(x, t; n) - \frac{\sigma_i^2}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t; n) = -\lambda_i p_i(x, t; n) + \lambda_i p_{1-i}(x - h_i, t; n-1),\ t > 0, \tag{2.4}$$

$$i = 0, 1,\ n \geq 1.$$

Moreover

$$p_i(x, t; 0) = \mathrm{e}^{-\lambda_i t}\psi_i(x, t), \tag{2.5}$$

where

$$\psi_i(x, t) = \frac{1}{\sigma_i\sqrt{2\pi t}}\mathrm{e}^{-\frac{(x-c_i t)^2}{2\sigma_i^2 t}}, \tag{2.6}$$

and

$$p_i(x, t; n)|_{t\downarrow 0} = 0,\ n \geq 1,\ i = 0, 1.$$

Proof. The equality (2.5) follows from definitions (2.2)-(2.3).

To derive (2.4) note that from the properties of Poisson and Wiener processes (see e.g. Protter (1990)) for any $t_2 > t_1$ it follows that

$$\mathcal{X}_i(t_2) = \mathcal{X}_i(t_1) + \mathcal{X}'_{\varepsilon_i(t_1)}(t_2 - t_1), \tag{2.7}$$

where $\mathcal{X}'_i$ is a copy of the process $\mathcal{X}_i$, $i = 0, 1$ which is independent of the original.

Let $\Delta t > 0$. From (2.7) it follows that $\mathcal{X}_i(t+\Delta t) = \mathcal{X}_i(\Delta t) + \mathcal{X}'_i(t)$. Let τ is the random variable uniformly distributed on $[0,\ \Delta t]$ and independent of $\mathcal{X}_i$. Notice that $\mathcal{X}_i(\Delta t) = c_i\Delta t + \sigma_i w(\Delta t)$, if $N_i(\Delta t) = 0$, and $\mathcal{X}_i(\Delta t) \overset{d}{=} c_i\tau + c_{1-i}(\Delta t - \tau) + \sigma_i w(\tau) + \sigma_{1-i}w(\Delta t - \tau) + h_i$, if $N_i(\Delta t) = 1$.

Since $\mathbb{P}_i(N_i(\Delta t) > 1) = o(\Delta t)$ as $\Delta t \to 0$, then conditioning on a jump in $(0, \Delta t)$ we have

$$p_i(x, t+\Delta t; n) = (1-\lambda_i\Delta t)p_i(\cdot, t; n)*\psi_i(\cdot, \Delta t)(x) + \lambda_i\Delta t p_{1-i}(\cdot, t; n-1)*\tilde{\psi}_i(\cdot, \Delta t)(x - h_i) + o(\Delta t), \tag{2.8}$$

$i = 0, 1,\ \Delta t \to 0$. Here $\psi_i(\cdot, \Delta t)$, the distribution density of $c_i\Delta t + \sigma_i w(\Delta t)$, is defined in (2.6), and $\tilde{\psi}_i(\cdot, \Delta t)$ is the distribution density of $c_i\tau + c_{1-i}(\Delta t - \tau) + \sigma_i w(\tau) + \sigma_{1-i}w(\Delta t - \tau)$; the notation $*$ is used for the convolution in spacial variables.

It is easy to see, that $\psi_i(x, \Delta t),\ \tilde{\psi}_i(x, \Delta t) \to \delta(x)$ as $\Delta t \to 0$. Hence

$$p_i(\cdot, t; n) * \psi_i(\cdot, \Delta t)(x) \to p_i(x, t; n),$$

$$p_{1-i}(\cdot, t; n-1) * \tilde{\psi}_i(\cdot, \Delta t)(x - h_i) \to p_{1-i}(x - h_i, t; n-1) \tag{2.9}$$

as $\Delta t \to 0$.

Then,

$$\frac{1}{\Delta t}\left[p_i(\cdot, t; n) * \psi_i(\cdot, \Delta t)(x) - p_i(x, t; n)\right] = \frac{1}{\Delta t}\left[\int_{-\infty}^{\infty} p_i(x - y, t; n)\psi_i(y, \Delta t)\mathrm{d}y - p_i(x, t; n)\right]$$

$$= \frac{1}{\Delta t}\int_{-\infty}^{\infty}\left[p_i(x - c_i\Delta t - y\sigma_i\sqrt{\Delta t}, t; n) - p_i(x, t; n)\right]\psi(y)\mathrm{d}y,$$

where $\psi = \psi(\cdot)$ is $\mathcal{N}(0,1)$-density. The latter value equals to

$$\frac{1}{\Delta t}\int_{-\infty}^{\infty}\psi(y)\left[\frac{\partial p_i}{\partial x}(x, t; n)(-c_i\Delta t - y\sigma_i\sqrt{\Delta t}) + \frac{1}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t; n)(-c_i\Delta t - y\sigma_i\sqrt{\Delta t})^2 + o(\Delta t)\right]\mathrm{d}y$$

$$= \frac{1}{\Delta t}\int_{-\infty}^{\infty}\psi(y)\left[\frac{\partial p_i}{\partial x}(x, t; n)(-c_i\Delta t) + \frac{1}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t; n)y^2\sigma_i^2\Delta t + o(\Delta t)\right]\mathrm{d}y$$

$$\to -c_i\frac{\partial p_i}{\partial x}(x, t; n) + \frac{\sigma_i^2}{2}\frac{\partial^2 p_i}{\partial x^2}(x, t; n),$$

so system (2.4) follows from (2.8) and (2.9). □

It is easy to solve system (2.4). Let us define functions $q_i = q_i(x, t; n)$. For $n \geq 1$

$$q_0(x, t; 2n) = \frac{\lambda_0^n\lambda_1^n}{(c_0 - c_1)^{2n}} \cdot \frac{(c_0t - x)^{n-1}(x - c_1t)^n}{(n-1)!n!}\theta(x, t),$$
$$q_1(x, t; 2n) = \frac{\lambda_0^n\lambda_1^n}{(c_0 - c_1)^{2n}} \cdot \frac{(c_0t - x)^n(x - c_1t)^{n-1}}{n!(n-1)!}\theta(x, t), \tag{2.10}$$

and for $n \geq 0$

$$q_0(x, t; 2n+1) = \frac{\lambda_0^{n+1}\lambda_1^n}{(c_0 - c_1)^{2n+1}} \cdot \frac{(c_0t - x)^n(x - c_1t)^n}{(n!)^2}\theta(x, t),$$
$$q_1(x, t; 2n+1) = \frac{\lambda_0^n\lambda_1^{n+1}}{(c_0 - c_1)^{2n+1}} \cdot \frac{(c_0t - x)^n(x - c_1t)^n}{(n!)^2}\theta(x, t). \tag{2.11}$$

Here $\theta(x, t) = \exp\left\{-\frac{\lambda_1}{c_0 - c_1}(c_0t - x) - \frac{\lambda_0}{c_0 - c_1}(x - c_1t)\right\}\mathbf{1}_{\{c_1t < x < c_0t\}}$.

The distribution densities $p_i^{(0)}$ of the jump telegraph process without a diffusion term can be expressed as follows. Resolving equation (2.4) with $\sigma_0 = \sigma_1 = 0$ we have

$$p_i^{(0)}(x, t; n) = q_i(x - j_i(n), t; n), \tag{2.12}$$

where $j_i(n) = [(n+1)/2]h_i + [n/2]h_{1-i},\ n = 0, 1, \ldots$. Equation (2.5) now means that $p_0^{(0)}(x,t;0) = \mathrm{e}^{-\lambda_0 t}\delta(x - c_0 t)$, $p_1^{(0)}(x,t;0) = \mathrm{e}^{-\lambda_1 t}\delta(x - c_1 t)$.

Conditioning on the number of switches we get the probability density of the jump telegraph process which is described by parameters $< c_0, \lambda_0, h_0 >$ and $< c_1, \lambda_1, h_1 >$:

$$p_i^{(0)}(x,t) = \sum_{n=0}^{\infty} p_i^{(0)}(x,t;n). \tag{2.13}$$

Remark 2.1. *Formula (2.13) in particular case $B = h_0 + h_1 = 0$ becomes*

$$p_i^{(0)}(x,t) = \mathrm{e}^{-\lambda_i t} \cdot \delta(x - c_i t)$$

$$+\frac{\theta(x,t)}{c_0 - c_1}\left[\lambda_i \exp\left(\frac{\lambda_0 - \lambda_1}{c_0 - c_1} h_i\right) I_0\left(2\frac{\sqrt{\lambda_0\lambda_1(c_0 t - x + h_i)(x - h_i - c_1 t)}}{c_0 - c_1}\right)\right.$$

$$\left. + \sqrt{\lambda_0\lambda_1}\left(\frac{x - c_1 t}{c_0 t - x}\right)^{\frac{1}{2} - i} I_1\left(2\frac{\sqrt{\lambda_0\lambda_1(c_0 t - x)(x - c_1 t)}}{c_0 - c_1}\right)\right],$$

where $I_0(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{(n!)^2}$ and $I_1(z) = I_0'(z)$ are modified Bessel functions. Compare with Beghin et al (2001).

We apply previous results to obtain the distributions of times which the process ε_i spends in the certain state.

Let $T_i = T_i(t) = \int_0^t \mathbf{1}_{\{\varepsilon_i(\tau)=0\}} \mathrm{d}\tau,\ i = 0, 1$ be the total time between 0 and t spending by the process ε_i in the state 0 starting form the state i.

If we consider a standard telegraph processes with velocities $c_0 = 1, c_1 = -1$, $\overline{\mathcal{T}_0}(t) = \int_0^t (-1)^{N_0(\tau)} \mathrm{d}\tau$ and $\overline{\mathcal{T}_1}(t) = -\int_0^t (-1)^{N_1(\tau)} \mathrm{d}\tau$, then

$$\overline{\mathcal{T}_0}(t) = T_0 - (t - T_0) = 2T_0 - t \quad \text{and} \quad \overline{\mathcal{T}_1}(t) = 2T_1 - t. \tag{2.14}$$

Let $f_i(\tau, t; n), 0 \le \tau \le t$ denote the density of T_i: for all measurable $\Upsilon \subset [0, t]$

$$\int_{\Upsilon} f_i(\tau, t; n) \mathrm{d}\tau = \mathbb{P}_i\{T_i \in \Upsilon,\ N_i(t) = n\} \tag{2.15}$$

Applying (2.14) we can notice that

$$f_0(\tau, t; n) = 2\bar{p}_0(2\tau - t, t; n), \qquad f_1(\tau, t; n) = 2\bar{p}_1(2\tau - t, t; n), \tag{2.16}$$

where $\bar{p}_0$ and $\bar{p}_1$ are the densities of the standard telegraph process $\overline{\mathcal{T}_0}$ and $\overline{\mathcal{T}_1}$. Functions $\bar{p}_0$ and $\bar{p}_1$ are defined in (2.10)-(2.12) with $c_0 = 1$, $c_1 = -1$ and $h_0 = h_1 = 0$.

Using formulae for densities $\bar{p}_i$, which are obtained in (2.10)-(2.12), from (2.16) we have

$$f_0(\tau, t; 0) = \mathrm{e}^{-\lambda_0 t}\delta(\tau - t), \quad f_1(\tau, t; 0) = \mathrm{e}^{-\lambda_1 t}\delta(\tau).$$

For $n \ge 1$

$$f_0(\tau, t; 2n) = \lambda_0^n \lambda_1^n \frac{(t-\tau)^{n-1}\tau^n}{(n-1)!n!} \mathrm{e}^{-\lambda_0\tau - \lambda_1(t-\tau)} \mathbf{1}_{\{0 \le \tau \le t\}}, \tag{2.17}$$

$$f_1(\tau, t; 2n) = \lambda_0^n \lambda_1^n \frac{(t-\tau)^{n}\tau^{n-1}}{(n-1)!n!} \mathrm{e}^{-\lambda_0\tau - \lambda_1(t-\tau)} \mathbf{1}_{\{0 \le \tau \le t\}}, \tag{2.18}$$

and for $n \geq 0$

$$f_0(\tau, t; 2n+1) = \lambda_0^{n+1}\lambda_1^n \frac{(t-\tau)^n\tau^n}{(n!)^2} \mathrm{e}^{-\lambda_0\tau-\lambda_1(t-\tau)} \mathbf{1}_{\{0\leq\tau\leq t\}}, \tag{2.19}$$

$$f_1(\tau, t; 2n+1) = \lambda_0^{n}\lambda_1^{n+1} \frac{(t-\tau)^n\tau^n}{(n!)^2} \mathrm{e}^{-\lambda_0\tau-\lambda_1(t-\tau)} \mathbf{1}_{\{0\leq\tau\leq t\}}. \tag{2.20}$$

Summarizing we have the following expressions for the densities $f_i(\tau, t)$ of the spending time of the the process $\varepsilon_i(\tau), 0 \leq \tau \leq t$ in state 0:

$$\begin{aligned} f_0(\tau, t) = \mathrm{e}^{-\lambda_0 t}\delta(\tau - t) + \mathrm{e}^{-\lambda_0\tau-\lambda_1(t-\tau)} &\Big[\lambda_0 I_0(2\sqrt{\lambda_0\lambda_1\tau(t-\tau)}) \\ &+\sqrt{\lambda_0\lambda_1}\sqrt{\frac{\tau}{t-\tau}} I_1(2\sqrt{\lambda_0\lambda_1\tau(t-\tau)})\Big] \mathbf{1}_{\{0\leq\tau\leq t\}}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} f_1(\tau, t) = \mathrm{e}^{-\lambda_1 t}\delta(\tau) + \mathrm{e}^{-\lambda_0\tau-\lambda_1(t-\tau)} &\Big[\lambda_1 I_0(2\sqrt{\lambda_0\lambda_1\tau(t-\tau)}) \\ &+\sqrt{\lambda_0\lambda_1}\sqrt{\frac{t-\tau}{\tau}} I_1(2\sqrt{\lambda_0\lambda_1\tau(t-\tau)})\Big] \mathbf{1}_{\{0\leq\tau\leq t\}}. \end{aligned} \tag{2.22}$$

In terms of $f_i(\tau, t)$ it is possible to express the distribution of the telegraph-diffusion process. If $T_i(t) = \int\limits_0^t \mathbf{1}_{\{\varepsilon_i(\tau)=0\}}\mathrm{d}\tau$, then $\mathcal{T}_i(t) = c_0 T_i(t) + c_1(t - T_i(t))$ and $\mathcal{D}_i(t) \overset{d}{=} \sigma_0 w(T_i(t)) + \sigma_1 w'(t - T_i(t))$, where w and w' are independent.

Let $a_\tau = c_0\tau + c_1(t-\tau)$ and $\Sigma_\tau^2 = \sigma_0^2\tau + \sigma_1^2(t-\tau)$. The distribution densities of telegraph-diffusion process $\mathcal{T}_i(t) + \mathcal{D}_i(t),\ t \geq 0$ can be expressed as follows:

$$p_i(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{f_i(\tau, t)}{\Sigma_\tau} \exp\left\{-\frac{1}{2\Sigma_\tau^2}(x - a_\tau)^2\right\} \mathrm{d}\tau.$$

Next, we describe in this framework martingales and martingale measures. The following theorem could be considered as a version of the Doob-Meyer decomposition for telegraph-diffusion processes with alternating intensities.

Theorem 2.2. *Jump telegraph-diffusion process* $\mathcal{T}_i + \mathcal{J}_i + \mathcal{D}_i, i = 0, 1$ *is a martingale if and only if* $c_0 = -\lambda_0 h_0$ *and* $c_1 = -\lambda_1 h_1$.

Proof. The processes $\sigma_{\varepsilon_i(t)}, t \geq 0, i = 0, 1$ are $\mathfrak{F}_t$-measurable. Hence the processes $\mathcal{D}_i = \mathcal{D}_i(t) = \int_0^t \sigma_{\varepsilon_i(\tau)}\mathrm{d}w(\tau),\ t \geq 0,\ i = 0, 1$ are $\mathfrak{F}_t$-martingales. Now, the result follows from Theorem 2.1 of Ratanov (2007). □

Let $h_0, h_1 > -1$. Denote

$$\kappa_i(t) = \prod_{k=1}^{N_i(t)} (1 + h_{\varepsilon_i(\tau_k-)}). \tag{2.23}$$

Corollary 2.1. *The process* $\exp\{\mathcal{T}_i(t) + \mathcal{D}_i(t)\}\kappa_i(t)$ *is a martingale if and only if* $c_i + \sigma_i^2/2 = -\lambda_i h_i,\ i = 0, 1.$

Proof. It is sufficient to notice that $\exp\{\mathcal{T}_i(t)+\mathcal{D}_i(t)\}\kappa_i(t) = \mathcal{E}_t(\mathcal{T}_i+\mathcal{J}_i+\mathcal{D}_i+1/2\int_0^t \sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau)$, where $\mathcal{E}_t(\cdot)$ denote a stochastic exponential (see Protter (1990)). The corollary follows from Theorem 2.2. □

Now we study the properties of jump telegraph-diffusion processes under a change of measure. Let $\mathcal{T}_i^*$, $i=0,1$ be the telegraph processes with states $< c_0^*, \lambda_0 >$ and $< c_1^*, \lambda_1 >$, and $J_i^* = -\sum\limits_{j=1}^{N_i(t)} c^*_{\varepsilon_i(\tau_j-)}/\lambda_{\varepsilon_i(\tau_j-)}$, $i=0,1$ be the jump processes with jump values $h_i^* = -c_i^*/\lambda_i > -1$, which let the sum $\mathcal{T}_i^* + \mathcal{J}_i^*$ to be a martingale. Let $\mathcal{D}_i^* = \int\limits_0^t \sigma^*_{\varepsilon_i(\tau)}\mathrm{d}w(\tau)$ be the diffusion with alternating diffusion coefficients $\sigma_i^*, i=0,1$. Consider a probability measure $\mathbb{P}_i^*$ with a local density with respect to $\mathbb{P}_i$:

$$Z_i(t) = \frac{\mathbb{P}_i^*}{\mathbb{P}_i}|_t = \mathcal{E}_t(\mathcal{T}_i^* + \mathcal{J}_i^* + \mathcal{D}_i^*) = \exp\left(\mathcal{T}_i^*(t) + \mathcal{D}_i^*(t) - \frac{1}{2}\int_0^t (\sigma^*_{\varepsilon_i(s)})^2\mathrm{d}s\right)\kappa_i^*(t), \quad (2.24)$$

where $\kappa_i^*(t)$ is defined in (2.23) with h_i^* instead of h_i.

Theorem 2.3 (Girsanov theorem)**.** *Under the probability measure* $\mathbb{P}_i^*$

1) process $\tilde{w}(t) := w(t) - \int\limits_0^t \sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau$ *is a standard Brownian motion;*

2) counting Poisson process $N_i(t)$ *has intensities* $\lambda_i^* := \lambda_i(1+h_i^*) = \lambda_i - c_i^*$.

Proof. Let $U_i(t) := \exp\{z\tilde{w}(t)\} = \exp\{z(w(t) - \int_0^t \sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau)\}$. For 1) it is sufficient to show that for any $t_1 < t$

$$\mathbb{E}_i\{Z_i(t)U_i(t) \mid \mathcal{F}_{t_1}\} = \mathrm{e}^{z^2(t-t_1)/2}Z_i(t_1)U_i(t_1).$$

We prove it for $t_1 = 0$ (see (2.7)).

Notice that

$$Z_i(t)U_i(t) = \exp\left\{\mathcal{T}_i^*(t) + \mathcal{D}_i^*(t) - \frac{1}{2}\int\limits_0^t (\sigma^*_{\varepsilon_i(\tau)})^2\mathrm{d}\tau + zw(t) - z\int\limits_0^t \sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau\right\}\kappa_i^*(t)$$

$$= \exp\left\{\int\limits_0^t\left(c_{\varepsilon_i(\tau)} - \frac{1}{2}{\sigma^*_{\varepsilon_i(\tau)}}^2 - z\sigma^*_{\varepsilon_i(\tau)}\right)\mathrm{d}\tau + \int\limits_0^t(\sigma^*_{\varepsilon_i(\tau)} + z)\mathrm{d}w(\tau)\right\}\kappa_i^*(t)$$

$$= \mathcal{E}_t\left(\mathcal{T}_i^* + \mathcal{D}_i^* + \mathcal{J}_i^* + zw\right)\exp(z^2t/2).$$

Thus $\mathbb{E}_i(Z_i(t)U_i(t)) = \exp(z^2t/2)$.

To prove the second part of the theorem we denote $\pi_i^*(t;n) = \mathbb{P}_i^*\{N_i(t) = n\} = \mathbb{E}_i(Z_i(t)\mathbf{1}_{\{N_i(t)=n\}}) = \kappa_i^*(n)\int_{-\infty}^{\infty}\mathrm{e}^x p_i^*(x,t;n)\mathrm{d}x$, where $\kappa_i^*(n) = \prod\limits_{k=1}^{n}(1+h_{\varepsilon_i(\tau_k-)})$, and $p_i^* = p_i^*(x,t;n)$ are (generalized) probability densities of telegraph-diffusion process $X_i^*(t) + D_i^*(t) - \int_0^t(\sigma^*_{\varepsilon_i(\tau)})^2\mathrm{d}\tau/2$. Notice that functions $p_i^*(x,t;n)$ satisfy system (2.4) with $c_i^* - (\sigma_i^*)^2/2$ and σ_i^* instead of c_i and σ_i respectively. Therefore

$$\frac{\mathrm{d}\pi_i^*(t;n)}{\mathrm{d}t} = (c_i^* - \lambda_i)\pi_i^*(t;n) + \lambda_i(1+h_i^*)\pi_{1-i}^*(t;n-1).$$

Next notice that $\lambda_i - c_i^* = \lambda_i + \lambda_i h_i^* := \lambda_i^*$ and, thus

$$\frac{\mathrm{d}\pi_i^*(t;n)}{\mathrm{d}t} = -\lambda_i^* \pi_i^*(t;n) + \lambda_i^* \pi_{1-i}^*(t;n-1).$$

The second part of the theorem now follows from (2.1). $\square$

3 Jump telegraph-diffusion model

Let $\varepsilon_i = \varepsilon_i(t) = 0, 1,\ t \geq 0$ be a Markov switching process defined in Section 2 which indicates the current market state.

Consider $\mathcal{T}_i$, $\mathcal{J}_i$ and $\mathcal{D}_i$, which are defined in (2.2). Assume that $h_0, h_1 > -1$. First, we define the market with one risky asset. Assume that the price of the risky asset which initially is at the state i, follows the equation

$$\mathrm{d}S(t) = S(t-)\mathrm{d}(\mathcal{T}_i(t) + \mathcal{J}_i(t) + \mathcal{D}_i(t)), \quad i = 0, 1.$$

As it is observed in Section 2,

$$S(t) = S_0 \mathcal{E}_t(\mathcal{T}_i + \mathcal{J}_i + \mathcal{D}_i) = S_0 \exp\left(\mathcal{T}_i(t) + \mathcal{D}_i(t) - \frac{1}{2}\int_0^t \sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau\right)\kappa_i(t). \tag{3.1}$$

Let $r_i, r_i \geq 0$ is the interest rate of the market which is at the state $i,\ i = 0, 1$. Let us consider the geometric telegraph process of the form

$$B(t) = \exp\left\{\mathcal{Y}_i(t)\right\}, \qquad \mathcal{Y}_i(t) = \int\limits_0^t r_{\varepsilon_i(\tau)}\mathrm{d}\tau. \tag{3.2}$$

as a numeraire.

The model (3.1)-(3.2) is incomplete. Due to simplicity of this model the set $\mathcal{M}$ of equivalent risk-neutral measures can be described in detail.

Let us define an equivalent measure $\mathbb{P}_i^*$ by means of the density $Z_i(t)$ (see (2.24)) with c_i^*, $h_i^* = -c_i^*/\lambda_i > -1$ and with arbitrary σ_i^*. Due to Theorem 2.3 $c_i^* = \lambda_i - \lambda_i^* < \lambda_i, i = 0, 1$.

Let $\theta_0, \theta_1 > 0$. We denote $c_0^* = \lambda_0 - \theta_0$, $c_1^* = \lambda_1 - \theta_1$, $h_0^* = -1 + \theta_0/\lambda_0$, $h_1^* = -1 + \theta_1/\lambda_1$, and we take arbitrary σ_0^*, σ_1^*. Due to Theorem 2.3, under the measure $\mathbb{P}_i^*$ the driving Poisson process $N_i(t)$ has intensities $\lambda_i^* = \theta_i, i = 0, 1$. The equivalent risk-neutral measures for the model (3.1)-(3.2) depend on two positive parameters θ_0 and θ_1.

Theorem 3.1. *Let probability measure $\mathbb{P}_i^*$ be defined by means of the density $Z_i(t), t \geq 0$. Let $\sigma_0 \neq 0$ and $\sigma_1 \neq 0$. The process $B(t)^{-1}S(t)$ is a $\mathbb{P}_i^*$-martingale if and only if the measure $\mathbb{P}_i^*$ is defined by parameters $c_0^* = \lambda_0 - \theta_0$, $c_1^* = \lambda_1 - \theta_1$, $h_0^* = -1 + \theta_0/\lambda_0$, $h_1^* = -1 + \theta_1/\lambda_1$ and σ_0^* and σ_1^* which are as follows: $\sigma_0^* = (r_0 - c_0 - h_0\theta_0)/\sigma_0$ and $\sigma_1^* = (r_1 - c_1 - h_1\theta_1)/\sigma_1$, $\theta_0, \theta_1 > 0$.*

Proof. Indeed,

$$Z_i(t)B(t)^{-1}S(t) = S_0 \exp\{Y_i(t)\}\tilde{\kappa}_i(t),$$

where

$$Y_i(t) = \mathcal{T}_i(t) + \mathcal{T}_i^*(t) + \mathcal{D}_i(t) + \mathcal{D}_i^*(t) - \frac{1}{2}\int_0^t ({\sigma_{\varepsilon_i(\tau)}}^2 + {\sigma^*_{\varepsilon_i(\tau)}}^2)\mathrm{d}\tau - \mathcal{Y}_i(t)$$

and $\tilde{\kappa}_i(t)$ is defined as in (2.23) with $\tilde{h}_i$ instead of h_i. Here $\tilde{h}_i$ satisfies the equation

$$1 + \tilde{h}_i = (1 + h_i^*)(1 + h_i), \;\; i = 0, 1.$$

Thus $\tilde{h}_i = h_i + h_i^* + h_i h_i^* = h_i + (-1+\theta_i/\lambda_i) + h_i(-1+\theta_i/\lambda_i) = \theta_i(1+h_i)/\lambda_i - 1, \; i = 0, 1$. Using Corollary 2.1 we see that $Z_i(t)B(t)^{-1}S(t)$ is the $\mathbb{P}_i$-martingale, if and only if

$$\begin{cases} c_0 + c_0^* - r_0 + \sigma_0\sigma_0^* = -\lambda_0\tilde{h}_0 \\ c_1 + c_1^* - r_1 + \sigma_1\sigma_1^* = -\lambda_1\tilde{h}_1 \end{cases}.$$

Note that $c_i^* = \lambda_i - \theta_i$ and $\lambda_i\tilde{h}_i = \theta_i(1 + h_i) - \lambda_i$, so

$$\begin{cases} c_0 + (\lambda_0 - \theta_0) - r_0 + \sigma_0\sigma_0^* = -\theta_0(1 + h_0) + \lambda_0 \\ c_1 + (\lambda_1 - \theta_1) - r_1 + \sigma_1\sigma_1^* = -\theta_1(1 + h_1) + \lambda_1 \end{cases},$$

and then

$$\begin{cases} c_0 - r_0 + \sigma_0\sigma_0^* = -\theta_0 h_0 \\ c_1 - r_1 + \sigma_1\sigma_1^* = -\theta_1 h_1 \end{cases}. \tag{3.3}$$

Therefore $\sigma_i^* = (r_i - c_i - h_i\theta_i)/\sigma_i, \; i = 0, 1.$ □

Remark 3.1. *The case of $\sigma_0 = \sigma_1 = 0$ is called jump-telegraph model, and it is complete. In this case the martingale measure is defined by $c_i^* = \lambda_i - \lambda_i^*$ and $\lambda_i^* = \frac{r_i - c_i}{h_i}$ as the new intensities of switchings. See Ratanov (2007) for details.*

The Black-Scholes model respects to $h_0 = h_1 = 0, \; \sigma_0 = \sigma_1 := \sigma, \; c_0 = c_1 := c, \; r_0 = r_1 = r$. In this case system (3.3) has the unique solution $\sigma_0^* = \sigma_1^* = \sigma^* = \frac{r-c}{\sigma}$. It means that the martingale measure is unique. Due to Girsanov theorem 2.3 the process $w(t) - \sigma^* t$ is Brownian motion under the new measure, which repeats the classic result.

To complete the model we add a new asset. Consider the market of two risky assets which are driven by common Brownian motion w and counting Poisson processes N_i:

$$\mathrm{d}S^{(1)}(t) = S^{(1)}(t-)\mathrm{d}(\mathcal{T}_i^{(1)}(t) + \mathcal{J}_i^{(1)}(t) + \mathcal{D}_i^{(1)}(t)), \tag{3.4}$$

$$\mathrm{d}S^{(2)}(t) = S^{(2)}(t-)\mathrm{d}(\mathcal{T}_i^{(2)}(t) + \mathcal{J}_i^{(2)}(t) + \mathcal{D}_i^{(2)}(t)). \tag{3.5}$$

As usual, $i = 0, 1$ denotes the initial market state.

Denote

$$\Delta_0^{(h)} = \begin{vmatrix} \sigma_0^{(1)} & h_0^{(1)} \\ \sigma_0^{(2)} & h_0^{(2)} \end{vmatrix} = \sigma_0^{(1)}h_0^{(2)} - \sigma_0^{(2)}h_0^{(1)}, \quad \Delta_1^{(h)} = \begin{vmatrix} \sigma_1^{(1)} & h_1^{(1)} \\ \sigma_1^{(2)} & h_1^{(2)} \end{vmatrix} = \sigma_1^{(1)}h_1^{(2)} - \sigma_1^{(2)}h_1^{(1)},$$

and

$$\Delta_0^{(r-c)} = \begin{vmatrix} \sigma_0^{(1)} & r_0 - c_0^{(1)} \\ \sigma_0^{(2)} & r_0 - c_0^{(2)} \end{vmatrix} = \sigma_0^{(1)}(r_0 - c_0^{(2)}) - \sigma_0^{(2)}(r_0 - c_0^{(1)}),$$

$$\Delta_1^{(r-c)} = \begin{vmatrix} \sigma_1^{(1)} & r_1 - c_1^{(1)} \\ \sigma_1^{(2)} & r_1 - c_1^{(2)} \end{vmatrix} = \sigma_1^{(1)}(r_1 - c_1^{(2)}) - \sigma_1^{(2)}(r_1 - c_1^{(1)}).$$

Let $\Delta_0^{(h)} \neq 0,\ \Delta_1^{(h)} \neq 0$. We assume that

$$\lambda_i^* := \frac{\Delta_i^{(r-c)}}{\Delta_i^{(h)}} > 0. \tag{3.6}$$

Theorem 3.2. *Both processes $B(t)^{-1}S^{(m)}(t), t \geq 0, m = 1,2$ are $\mathbb{P}_i^*$-martingales if and only if the measure $\mathbb{P}_i^*$ is defined by (2.24) with the following parameters:*

$$\sigma_0^* = \frac{(r_0 - c_0^{(1)})h_0^{(2)} - (r_0 - c_0^{(2)})h_0^{(1)}}{\Delta_0^{(h)}}, \qquad \sigma_1^* = \frac{(r_1 - c_1^{(1)})h_1^{(2)} - (r_1 - c_1^{(2)})h_1^{(1)}}{\Delta_1^{(h)}}, \tag{3.7}$$

$$c_0^* = \lambda_0 - \frac{\Delta_0^{(r-c)}}{\Delta_0^{(h)}}, \qquad c_1^* = \lambda_1 - \frac{\Delta_1^{(r-c)}}{\Delta_1^{(h)}} \tag{3.8}$$

and

$$h_0^* = -c_0^*/\lambda_0, \qquad h_1^* = -c_1^*/\lambda_1.$$

Under the measure $\mathbb{P}_i^*$ the rate of leaving the state i equals to λ_i^* defined in (3.6).

Proof. First notice

$$\begin{aligned} Z_i(t)B(t)^{-1}S^{(m)}(t) &= S^{(m)}(0)\mathcal{E}_t \exp(\mathcal{T}_i^* + \mathcal{J}_i^* + \mathcal{D}_i^*)\exp(-Y_i(t))\mathcal{E}_t(\mathcal{T}_i^{(m)} + \mathcal{J}_i^{(m)} + \mathcal{D}_i^{(m)}) \\ &= \exp\left(\mathcal{T}_i^*(t) + \mathcal{D}_i^*(t) - \frac{1}{2}\int_0^t {\sigma_{\varepsilon_i(\tau)}^*}^2 \mathrm{d}\tau\right)\kappa_i^*(t) \\ &\times \exp\left(\mathcal{T}_i^{(m)}(t) + \mathcal{D}_i^{(m)}(t) - Y_i(t) - \frac{1}{2}\int_0^t {\sigma_{\varepsilon_i(\tau)}^{(m)}}^2 \mathrm{d}\tau\right)\kappa_i^{(m)}(t) \\ &= \mathcal{E}_t\left(\mathcal{T}_i^{(m)} + \mathcal{T}_i^* + \mathcal{D}_i^{(m)} + \mathcal{D}_i^* - Y_i + \int_0^t \sigma_{\varepsilon_i(\tau)}^{(m)}\sigma_{\varepsilon_i(\tau)}^* \mathrm{d}\tau\right)\kappa_i^{(m)}(t)\kappa_i^*(t). \end{aligned}$$

Thus $Z_i(t)B(t)^{-1}S^{(m)}(t)$ is a martingale if and only if (Theorem 2.2)

$$\begin{cases} c_i^{(1)} + c_i^* - r_i + \sigma_i^{(1)}\sigma_i^* = -\lambda_i(h_i^{(1)} + h_i^* + h_i^{(1)}h_i^*) \\ c_i^{(2)} + c_i^* - r_i + \sigma_i^{(2)}\sigma_i^* = -\lambda_i(h_i^{(2)} + h_i^* + h_i^{(2)}h_i^*) \end{cases}. \tag{3.9}$$

Now using the identities $c_i^* = -\lambda_i h_i^*, i = 0,1$ we simplify the system (3.9) to

$$\begin{cases} \sigma_i^{(1)}\sigma_i^* - h_i^{(1)}c_i^* = r_i - c_i^{(1)} - \lambda_i h_i^{(1)} \\ \sigma_i^{(2)}\sigma_i^* - h_i^{(2)}c_i^* = r_i - c_i^{(2)} - \lambda_i h_i^{(2)} \end{cases}. \tag{3.10}$$

Systems (3.10) have the solutions described in (3.7)-(3.8).

Note that as it follows from Girsanov theorem, the intensity parameters under measure $\mathbb{P}_i^*$, λ_0^* and λ_1^* are defined in (3.6). ∎

Corollary 3.1. *Let $\Delta_0^{(h)} \neq 0,\ \Delta_1^{(h)} \neq 0$ and (3.6) is fulfilled. If the prices $S_i^{(1)}$ and $S_i^{(2)}$ of both risky assets are are defined in (3.4)-(3.5) with nonzero jumps, $h_0^{(m)} \neq 0, h_1^{(m)} \neq 0,\ m = 1, 2$, then*

$$\sigma_0^* = \frac{\alpha_0^{(1)} - \alpha_0^{(2)}}{\beta_0^{(1)} - \beta_0^{(2)}}, \qquad \sigma_1^* = \frac{\alpha_1^{(1)} - \alpha_1^{(2)}}{\beta_1^{(1)} - \beta_1^{(2)}}$$

and

$$c_0^* = \lambda_0 - \frac{\beta_0^{(1)}\alpha_0^{(2)} - \beta_0^{(2)}\alpha_0^{(1)}}{\beta_0^{(1)} - \beta_0^{(2)}}, \qquad c_1^* = \lambda_1 - \frac{\beta_1^{(1)}\alpha_1^{(2)} - \beta_1^{(2)}\alpha_1^{(1)}}{\beta_1^{(1)} - \beta_1^{(2)}},$$

where

$$\alpha_0^{(m)} = \frac{r_0 - c_0^{(m)}}{h_0^{(m)}}, \qquad \alpha_1^{(m)} = \frac{r_1 - c_1^{(m)}}{h_1^{(m)}}, \qquad \beta_0^{(m)} = \frac{\sigma_0^{(m)}}{h_0^{(m)}}, \qquad \beta_1^{(m)} = \frac{\sigma_1^{(m)}}{h_1^{(m)}}, \qquad m = 1, 2.$$

Remark 3.2. *If $\Delta_0^{(h)} = \Delta_1^{(h)} = 0$, then the system (3.10) does not have a solution (if $\Delta_0^{(r-c)} \neq 0,\ \Delta_1^{(r-c)} \neq 0$) or it has infinitely many solutions (if $\Delta_0^{(r-c)} = \Delta_1^{(r-c)} = 0$). It means arbitrage or incompleteness respectively.*

In particular case of the market model without jumps, i. e. $h_i^{(1)} = h_i^{(2)} = 0, i = 0, 1$, the market of two assets is arbitrage-free (respectively, the system (3.10) has solutions) if and only if the assets are similar:

$$\frac{r_i - c_i^{(1)}}{\sigma_i^{(1)}} = \frac{r_i - c_i^{(2)}}{\sigma_i^{(2)}} = \sigma_i^*,\ \ i = 0, 1.$$

In this case the model is incomplete.

Remark 3.3. *Hidden Markov model with $h_0^{(1)} = h_1^{(1)} = 0$ can be completed by adding a security that pays one unit of bond at the next time that the Markov chain $\varepsilon_i(t)$ changes state (see Guo (2001)). That change-of-state contract then becomes worthless and a new contract is issued that pays at the next change of state, and so on. Under natural pricing, this completes the model, and $\lambda_i^* = \frac{r_i\lambda_i}{r_i + k_i}$, where k_i is given, and can be thought as a risk-premium coefficient.*

Theorem 3.1 presents the unique risk-neutral measure for this completion of the market. It is given by (2.24) with $c_i^* = \lambda_i - \lambda_i^* = \frac{\lambda_i k_i}{r_i + k_i}$, $h_i^* = -1 + \lambda_i^*/\lambda_i = -\frac{k_i}{r_i + k_i}$ and $\sigma_i^* = (r_i - c_i)/\sigma_i,\ i = 0, 1$.

In our framework the stock (without jump component)

$$S^{(1)}(t) = S^{(1)}(0)\mathrm{e}^{\mathcal{T}_i(t) + \mathcal{D}_i(t) - \frac{1}{2}\int_0^t \sigma^2_{\varepsilon_i(\tau)}\mathrm{d}\tau}$$

can be naturally accompanied with the security which magnifies its value with the fixed rate at each moment of the change of state:

$$S^{(2)}(t) = \prod_{k=1}^{N_i(t)} (1 + h_{\varepsilon_i(\tau_k -)}), \qquad h_0, h_1 > 0.$$

This security can be considered as an insurance contract, different from change-of-state contract proposed in Guo (2001), that compensates losses provoked by state changes.

By the definition we see that $\Delta_i^{(h)} = \sigma_i h_i$, $\Delta_i^{(r-c)} = \sigma_i r_i$. Thus Theorem 3.2 gives $\lambda_i^* = r_i/h_i$, $\sigma_i^* = (r_i - c_i)/\sigma_i$, $c_i^* = \lambda_i - r_i/h_i$, $h_i^* = -1 + r_i/(\lambda_i h_i)$.

In contrast with Guo (2001) the security which completes the market model is perpetual, i. e. it not becomes worthless at the switching times.

Assume now that $\Delta_0^{(h)} \neq 0$ and $\Delta_1^{(h)} \neq 0$, and (3.6) is fulfilled. Therefore the market model can be completed. Let us present the formula for the price of standard call option. Let Z be a r.v. with normal distribution $\mathcal{N}(0, \sigma^2)$. We denote

$$\varphi(x, K, \sigma) := \mathbb{E}[x\mathrm{e}^{Z-\sigma^2/2} - K]^+ = xF(\frac{\ln(x/K) + \sigma^2/2}{\sigma}) - KF(\frac{\ln(x/K) - \sigma^2/2}{\sigma}), \quad (3.11)$$

where $F(x)$ is the distribution function of standard normal law:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} \mathrm{e}^{-y^2/2}\mathrm{d}y.$$

Let the market contains two risky assets (3.4)-(3.5). Consider the standard call option on the first asset with the claim $\big(S^{(1)}(T) - K\big)^+$. Therefore the call-price is

$$\mathfrak{c}_i = \mathbb{E}_i^*\{B(T)^{-1}(S_i^{(1)}(T) - K)^+\}, \quad (3.12)$$

if the market is starting with the state i. Here $\mathbb{E}_i^*$ is the expectation with respect to the martingale measure $\mathbb{P}_i^*$ which is constructed in Theorem 3.2.

By Girsanov theorem 2.3 the process $\tilde{w}(t) = w(t) - \int_0^t \sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau$ is the Brownian motion under new measure $\mathbb{P}_i^*$. Hence

$$B(T)^{-1}S^{(1)}(T) = S^{(1)}(0)\exp\left\{\mathcal{T}_i^{(1)}(T) + \int_0^T \sigma^{(1)}_{\varepsilon_i(\tau)}\mathrm{d}w(\tau) - \frac{1}{2}\int_0^T {\sigma^{(1)}_{\varepsilon_i(\tau)}}^2\mathrm{d}\tau - \mathcal{Y}_i(T)\right\}\kappa_i^{(1)}(T)$$

$$= S^{(1)}(0)\exp\left\{\mathcal{T}_i^{(1)}(T) + \int_0^T \sigma^{(1)}_{\varepsilon_i(\tau)}\mathrm{d}\tilde{w}(\tau) + \int_0^T \sigma^{(1)}_{\varepsilon_i(\tau)}\sigma^*_{\varepsilon_i(\tau)}\mathrm{d}\tau - \frac{1}{2}\int_0^T {\sigma^{(1)}_{\varepsilon_i(\tau)}}^2\mathrm{d}\tau - \mathcal{Y}_i(T)\right\}\kappa_i^{(1)}(T).$$

The first equation of (3.10) can be transformed to $c_i^{(1)} - r_i + \sigma_i^{(1)}\sigma_i^* = h_i^{(1)}(c_i^* - \lambda_i)$. From Girsanov theorem 2.3 we have $c_i^* - \lambda_i = -\lambda_i^*$. Let us introduce the telegraph process $\overline{\mathcal{T}_i^{(1)}}$ independent of $\tilde{w}$ which is driven by Poisson process with parameters λ_i^* and with the velocities $\tilde{c}_i = c_i^{(1)} - r_i + \sigma_i^{(1)}\sigma_i^* = -\lambda_i^* h_i^{(1)}$, $i = 0, 1$. So the martingale $B(T)^{-1}S^{(1)}(T)$ takes the form

$$B(T)^{-1}S^{(1)}(T) = S^{(1)}(0)\exp\left\{\overline{\mathcal{T}_i^{(1)}}(T) + \int_0^T \sigma^{(1)}_{\varepsilon_i(\tau)}\mathrm{d}\tilde{w}(\tau) - \frac{1}{2}\int_0^T {\sigma^{(1)}_{\varepsilon_i(\tau)}}^2\mathrm{d}\tau\right\}\kappa_i^{(1)}(T)$$

Again applying the property (2.7), from (3.12) we obtain

$$\mathfrak{c}_i = \int\limits_0^T \sum_{n=0}^{\infty} f_i(t, T; n)\varphi(x_i(t, T, n), K\mathrm{e}^{-r_0 t - r_1(T-t)}, \sqrt{\sigma_0^2 t + \sigma_1^2(T-t)})\mathrm{d}t, \; i = 0, 1. \quad (3.13)$$

Here $x_i(t, T, n) = S^{(1)}(0)\kappa_{i,n}\mathrm{e}^{\tilde{c}_0 t + \tilde{c}_1(T-t)}$ and

$$\kappa_{i,2n} = (1 + h_0^{(1)})^n(1 + h_1^{(1)})^n, \; i = 0, 1,$$

$$\kappa_{0,2n+1} = (1+h_0^{(1)})^{n+1}(1+h_1^{(1)})^n, \qquad \kappa_{1,2n+1} = (1+h_1^{(1)})^{n+1}(1+h_0^{(1)})^n,$$
$$n = 0, 1, 2, \ldots;$$

$f_i(t, T; n)$ are defined in (2.17)-(2.20) with $\lambda_0^* = \Delta_0^{(r-c)}/\Delta_0^{(h)}$, and $\lambda_1^* = \Delta_1^{(r-c)}/\Delta_1^{(h)}$ instead of λ_0 and λ_1; $\varphi(x, K, \sigma)$ is defined in (3.11). Notice that as in jump-telegraph model (see Ratanov (2007)) the option price (3.13) does not depend on λ_0 and λ_1.

In particular, if $h_0^{(1)} = h_1^{(1)} = 0$ and, nevertheless, $\Delta_0^{(h)} \neq 0$, $\Delta_1^{(h)} \neq 0$, we can summarize in (3.13) applying (2.17)-(2.20):

$$\mathfrak{c}_i = \int_0^T f_i(t, T)\varphi(S_0, K\mathrm{e}^{-r_0 t - r_1(T-t)}, \sqrt{\sigma_0^2 t + \sigma_1^2(T-t)})\mathrm{d}t, \ i = 0, 1,$$

where $f_i(t, T)$ are defined in (2.21) and (2.22) (cf. Guo (2001)).

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